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Exam I, Math 530, Fall 2014

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QUESTION 1. NOTE THAT THE GOAL of IN CLASS EXAM is to make sure that students know how to make use of class results, theorems and so on.

(i) Let G be an abelian group such that $|G| = n$ where $n \neq p^m$ for some prime p of R. Given that for every proper divisor of n, say d (note $d \neq n$), G has a cyclic subgroup of order d. Prove that for every proper divisor d of n, G has a unique subgroup of order d.

solution. Since $n \neq p^m$, we conclude that $n = ij$ such that $i, j \neq 1$ and gcd (i, j) = 1. Thus G has cyclic subgroups of order i and of order j. Thus G has an element, say a, of order i and G has an element, say b, of order j. Since $ab = ba$ and $gcd(i, j) = 1$, we know that the order of $ab = ij$. Thus G is cyclic. Hence we know that for every proper divisor d of n, G has a unique subgroup of order d.

(ii) Give me an example of an abelian group G such that $|G| = n = p^m$ for some prime p and for every proper divisor of n, say d ($d \neq 1$, $d \neq n$), G has at least two cyclic subgroups of order d.

solution. Let $n = 3^2$ and $G = (Z_3, +) \times (Z_3, +)$. Then $d = 3$ is the only proper divisor of 9 ($d \neq 1, d \neq 9$) and G has at least 2 distinct subgroups, namely: $Z_3 \times \{0\}$ and $\{0\} \times Z_3$

(iii) Let G be a group of order p^3 for some prime p. Prove that G is either abelian or $|C(G)| = p$ (note that $C(G)$ denotes the center of G).

Solution. By Cauchy we know $|C(G)| = p$ or p^2 or p^3. If $|C(G)| = p^3$, then G is an abelian group. Assume $|C(G)| = p^2$. Then we know that $G/C(G)$ is cyclic (since $|G/C(G)| = p$), hence we know that G must be abelian, and thus $|C(G)| = p^3$, a contradiction. Thus if G is not abelian, then $|C(G)| = p$

(iv) Let G be a group of order p^3 for some prime p. Prove that G is abelian or G has a normal proper subgroup D of G such that G/D is abelian. [Hint: you may use (ii)].

solution. By (ii), we know that if G is not abelian, then $|C(G)| = p$. Since $|G/C(G)| = p^2$, we know that G/C(G) is abelian.

(v) Let $(G, *)$ be a group and (D, δ) be an abelian group. Given f is a group homomorphism from G into D. Let $a, b \in G$. Prove that $a * b = b * a$ or $(b * a)^{-1} * (a * b) \in Ker(f)$.

Solution. Let $a, b \in G$. Assume that $a * b \neq b * a$. Since D is abelian, $f(a * b) = f(a)\delta f(b) = f(b)\delta f(a) = f(b * a)$. Since $f(a * b) = f(b * a)$, we have $f(b * a)^{-1}\delta f(a * b) = e_D$. We know $f(b * a)^{-1} = f((b * a)^{-1})$. Thus $f((b * a)^{-1} * (a * b)) = e_D$. Hence $(b * a)^{-1} * (a * b) \in Ker(f)$.

(vi) Let G be a group of order 9. Given $f : (Z_{45}, +) \to G$ is a surjective (onto) group homomorphism. Prove that G is group-isomorphic to $(Z_9, +)$ Find $ker(f)$. Given $f(2) = b \in G$. Let $S = \{a \in Z_{45} | f(a) = b\}$. Find the set S.

Solution. Since $Z_{45}/Ker(f)$ is group-isomorphic to G and Z_{45} is cyclic, we conclude that $Z_{45}/Ker(f)$ is cyclic and hence G is cyclic. Since Since $Z_{45}/Ker(f)$ is group-isomorphic to G and $|G| = 9$. We conclude $|Ker(f)| = 5$. Since Z_{45} is cyclic, it has exactly one subgroup of order 5. Thus $Ker(f) = \{0, 9, 18, 27, 36\}$. Let $K : Z_{45}/Ker(f) \to G$ such that $K(a + Ker(f)) = f(a)$. Then we know that K is an isomorphism. Since $f(2) = b$, we have $K(2 + Ker(f)) = f(2) = b$. Thus $S = 2 + Ker(f) = \{2, 11, 20, 29, 38\}$

(vii) Let G be a group of order n. Let M be the set of all non-isomorphic-groups of order n. Prove that M is a finite set.

Solution. Let D be a group of order n. Then we know that D is group-isomorphic to a subgroup of S_n. Since S_n has finitely many subgroups of order n (since S_n is a finite group), we conclude that the number of all non-isomorphic groups of order n is a finite number.

(viii) Suppose $|G| = 22$ such that G has a subgroup of order 11 and a normal subgroup of order 2. Without using Sylows Theorem, prove that G is group-isomorphic to $(Z_{22}, +)$.

Solution. Let D be a subgroup of G of order 11. Since $[G : D] = |G|/|D| = 2$ (i.e. D has only two left cosets), we conclude that D is a normal subgroup of G of order 11. Let H be a normal subgroup of G of order 2. Then $H \cap D = \{e\}$. Let $f : G \to G/D \times G/H$ such that $f(a) = (a * D, a * H)$. We know that G is an isomorphism. Since $|G/D| = 2$ and $|G/H| = 11$, we conclude that $G/D \equiv Z_2$ is cyclic and $G/H \equiv Z_{-11}$ is cyclic. Thus $G \equiv Z_2 \times Z_{11} \equiv Z_{22}$.

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